A new strategy adjustment process for computing a Nash equilibrium in a noncooperative more-person game

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A NEW STRATEGY-ADJUSTMENT PROCESS FOR COMPUTING A NASH EQUILIBRIUM IN A NONCOOPERATIVE MORE-PERSON GAME

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ABSTRACT

In this paper we present a strategy-adjustment process for finding a Nash equilibrium in a noncooperative N-person game. The adjustment consists of increasing the probabilities with which profitable strategies are played, while probabilities belonging to the unprofitable strategies are decreased. These increases and decreases are relative to the initial probabilities. This memorizing of the starting point is such that it prevents the process from both cycling and leaving the strategy space. Besides, the process converges under very weak conditions and can be followed discretely by a so-called simplicial algorithm. Finally, the process is compared with other adjustment processes on the strategy space.

1. Introduction

Recently, van der Laan and Talman (4) developed several adjustment processes for solving the nonlinear complementarity problem (NLCP) on the n-dimensional unit simplex. This problem concerns the search for a vector x^* in $S^n = \{x \in \mathbb{R}_+^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1\}$ for which $z(x^*) \leq 0$, where z is a continuous function from S^n to $\mathbb{R}^{n+1}$ satisfying $x^\top z(x) = 0$ for all x in S^n. The processes mentioned can start anywhere and converge to a solution when z is continuously differentiable. Besides, the processes can be followed discretely and arbitrary close by so-called simplicial algorithms. The latter feature provides the possibility of calculating a solution vector.

One of the main applications of the NLCP on S^n is the problem of finding an equilibrium price vector in a pure exchange economy. In that case S^n is regarded as the price space of the economy and the adjustment process can be interpreted as a price-adjustment process. In this way we can compare the adjustment processes on S^n of van der Laan and Talman with other price-adjustment processes like the Walras tatonnement process and the quasi-Newton processes of Smale (5). The Walras process only converges under very strong conditions, such as revealed preferences or gross-substitutability, while the Smale processes only converge locally. Besides, it is difficult to follow these processes discretely. All this gives reason to prefer the processes described in (4). This the more because the latter processes also posses an attractive economic interpretation. One of these processes generates a path of prices by decreasing initially the prices corresponding to goods with excess supply while those of the goods with excess demand are increased. Contrary to the Walras tatonnement process both the increases and decreases are relative to the initial price vector. This memorizing of the starting vector during the process prevents the process from both cycling and leaving the price simplex. The other processes in (4) only adapt the prices of the goods with a maximal excess supply or demand. Again the memorizing of the initial price vector guarantees convergency.

The generalization of these processes in order to solve the NLCP on the product space S of N unit simplices has been given by van den Elzen, van der Laan and Talman in (2). Let $S = \prod_{j=1}^{N} S^{n_j}$, so that x in S is equal to $x = (x_1^T, \ldots, x_N^T)^T$ with $x_j \in S^{n_j}$, $j \in I_N = \{1,2,\ldots,N\}$. In the NLCP on S we search for a vector x^* in S for which $z(x^*) \leq 0$, where $z\colon S \to \mathbb{R}^{N+M}$, $M = \sum_{j=1}^{N} n_j$, is a continuous function satisfying $x_j^T z_j(x) = 0$ for all $x \in S$ and $j \in I_N$. In (2) three processes were introduced, the sum-, the product- and the exponent-process. The

latter process can be considered as the generalization of the Walrasian type adjustment process in (4) while the sum- and the product-process are based on the other two processes on S^n. Furthermore, these processes on S posses the same nice properties as their counterparts on S^n.
Problems which can be transformed into an NLCP on S are the problem of finding a Nash equilibrium in a noncooperative game and the search for an equilibrium price vector in an international trade model with internationally traded common goods and domestic goods traded only in one country. The first problem can be found in (1) and the second one in (3). When applied to a noncooperative game the processes become strategy-adjustment processes while in case of an economy we can interprete them as price-adjustment processes. The main subject of this paper concerns the interpretation of the exponent-process as a strategy-adjustment process. This interpretation is given in section 4. In that section we also compare the exponent-process with the sum- and the product-process when these processes are applied to a noncooperative game. Section 2 deals with the transformation of the problem of finding a Nash equilibrium in a noncooperative N-person game into an NLCP on S. A mathematical treatment of the exponent-process for solving the general NLCP on S is given in section 3.

2. The noncooperative N-person game in NLCP-form

In this section we transform the problem of finding a Nash equilibrium strategy vector in a noncooperative game into an NLCP on S. We follow the description as given in (1). The noncooperative N-person game consists of N players indexed with j, $j \in I_N$, where player j possesses n_j+1 pure strategies denoted by (j,k), $k = 1,\ldots,n_j+1$. In the sequel the set of pure strategies of player j is denoted by $I(j)$, i.e. $I(j) = \{(j,1), (j,2),\ldots,(j,n_j+1)\}$ while $I = \bigcup_{j=1}^{N} I(j)$ is the set of all pure strategies in the game. The pure strategy vector $\underline{k} = ((1,k_1),\ (2,k_2),\dots,(N,k_N))^\top \in \prod_{j=1}^{N} I(j)$ denotes that player j plays his k_j-th pure strategy. The set J is the collection of all pure strategy vectors. Further, for each $j \in I_N$ there is a loss-function $a^j\colon J \to \mathbb{R}_+$, where $a^j(\underline{k})$ is the loss to player j if the pure strategy vector $\underline{k} \in J$ is played. A vector $x = (x_1^\top,\dots,x_N^\top)^\top$ in $S = \prod_{j=1}^{N} S^{n_j}$ is the mixed strategy vector indicating that player j plays his h-th strategy with probability x_{jh}, $(j,h) \in I(j)$ and $j \in I_N$. We call S the strategy space of the game. Notice that $\sum_{h=1}^{n_j+1} x_{jh} = 1$ and $x_{jh} \geq 0$ for all $h \in \{1,\dots,n_j+1\}$ so that x_j is indeed a probability vector for all $j \in I_N$. Furthermore, each pure strategy vector coincides with a vertex of S.

The expected loss to player j when strategy vector x in S is played is equal to

$$p^j(x) = \sum_{\underline{k}\in J} a^j(\underline{k}) \prod_{i=1}^{N} x_{ik_i}.$$

We call a strategy vector a Nash equilibrium if no player has incentives to change his strategy. Therefore we consider $m_h^j(x)$, being the marginal loss to player j if he plays his h-th pure strategy while the other players do not change their strategy x_i, $i \neq j$, i.e.

$$m_h^j(x) = \sum_{\substack{k_j=h\\ \underline{k}\in J}} a^j(\underline{k}) \prod_{\substack{i\neq j\\ i\in I_N}} x_{ik_i}.$$

In these terms a mixed strategy vector x^* in S is a Nash equilibrium if $p^j(x^*) - m_h^j(x^*) \leq 0$ for all $(j,h) \in I$. When the latter holds no change in a strategy of a player causes a decrease in his expected loss. With respect to the NLCP on S it is natural to take as the function z the continuous function from S to $\mathbb{R}^{N+M}$ defined by $z_{jh}(x) = p^j(x) - m_h^j(x)$, $(j,h) \in I$. In the sequel we often refer to this function as the excess-profit-function. Note that when $z_{jh}(x) > 0$ it is profitable

for player j to play his h-th pure strategy. Therefore we call a strategy (j,h) in I profitable to player j when strategy vector x in S is valid if $z_{jh}(x) > 0$, and unprofitable to player j if $z_{jh}(x) < 0$. Further, a strategy (j,h) in I is said to be in equilibrium at vector x when $z_{jh}(x) = 0$, while a player j, $j \in I_N$, is defined to be in equilibrium at x if $z_j(x) \leq 0$.
Because $p^j(x) = \sum_{h=1}^{n_j+1} x_{jh}\, m_h^j(x)$, we can derive that $x_j^T z_j(x) = 0$ for all $j \in I_N$ and $x \in S$. So, the problem of finding a Nash equilibrium in a noncooperative N-person game has been transformed into a problem of searching for a vector x^* in S for which $z(x^*) \leq 0$, where z is a continuous function from S to $\mathbb{R}^{N+M}$ satisfying $x_j^T z_j(x) = 0$, $x \in S$ and $j \in I_N$. Note that the excess-profit-function is multi-linear and therefore continuously differentiable, which will guarantee the convergence of the sum-, product- and exponent-process.

3. Mathematical exposure of the exponent-process on S

Here we present the exponent-process for solving the general NLCP on S. For a thorough treatment we refer the reader to (2).
The process is completely governed by the sign pattern of the function values and the location in S of the starting point. In fact the process generates vectors x in S such that for some $0 \leq b \leq 1$ and $\lambda_j \geq 0$, $j \in I_N$,

$$
\begin{aligned}
x_{jk} &= (1+\lambda_j)v_{jk} && \text{if } v_{jk} > 0 \text{ and } z_{jk}(x) > 0\\
x_{jk} &= \lambda_j && \text{if } v_{jk} = 0 \text{ and } z_{jk}(x) > 0\\
x_{jk} &= bv_{jk} && \text{if } z_{jk}(x) < 0\\
bv_{jk} \leq x_{jk} &\leq (1+\lambda_j)v_{jk} && \text{if } v_{jk} > 0 \text{ and } z_{jk}(x) = 0\\
0 \leq x_{jk} &\leq \lambda_j && \text{if } v_{jk} = 0 \text{ and } z_{jk}(x) = 0,
\end{aligned}
$$

where $v = (v_1^T, v_2^T, \ldots, v_N^T)^T$ is the arbitrarily chosen starting vector in S.
To describe the process in more detail, let the vector $s = (s_1^T, s_2^T, \ldots, s_N^T)^T$ be a sign vector in $\mathbb{R}^{N+M}$, i.e. $s_{jk} \in \{-1,0,+1\}$ for all $(j,k) \in I$. For each sign vector s in $\mathbb{R}^{N+M}$ we define for $j = 1,2,\ldots,N$,

$$I_j^-(s) = \{(j,k) \in I(j) \mid s_{jk} = -1\}$$

$$I_j^0(s) = \{(j,k) \in I(j) \mid s_{jk} = 0\}$$

$$I_j^+(s) = \{(j,k) \in I(j) \mid s_{jk} = +1\},$$

and

$$I^-(s) = \bigcup_j I_j^-(s),\ I^0(s) = \bigcup_j I_j^0(s) \text{ and } I^+(s) = \bigcup_j I_j^+(s),$$

where the union is over all $j \in I_N$. Furthermore we denote for all $j \in I_N$,

$$V_j = \{(j,k) \in I(j) \mid v_{jk} = 0\} \text{ and } V_j^c = I(j)\setminus V_j.$$

For each sign vector s we define the set $A(s)$ by

$$
\begin{aligned}
A(s) = \{x \in S \mid x_{jk} &= (1+\lambda_j)v_{jk} && \text{if } s_{jk} = +1 \text{ and } v_{jk} > 0\\
x_{jk} &= \lambda_j && \text{if } s_{jk} = +1 \text{ and } v_{jk} = 0\\
x_{jk} &= bv_{jk} && \text{if } s_{jk} = -1\\
bv_{jk} \le x_{jk} &\le (1+\lambda_j)v_{jk} && \text{if } s_{jk} = 0 \text{ and } v_{jk} > 0\\
0 \le x_{jk} &\le \lambda_j && \text{if } s_{jk} = 0 \text{ and } v_{jk} = 0\\
&\text{with } 0 \le b \le 1 \text{ and } \lambda_j \ge 0 \text{ for all } j \in I_N\}.
\end{aligned}
$$

Note that the sets $A(s)$ are related to the position of x in S with respect to the starting vector. Related to the sign pattern of the function values we define for each sign vector s the region C(s) by

$$C(s) = C\ell\{x \in S \mid \text{sgn } z(x) = s\},$$

where $C\ell(W)$ denotes the closure of the set W. The exponent-process now generates for varying s, vectors x in S lying in both A(s) and C(s). In (2) it is proved that only intersections related to sign vectors in the set τ^3 are relevant, where

$$\tau^3 = \{s \in \mathbb{R}^{N+M} \mid \forall_j \in I_N, \text{ either } I_j^+(s) = \phi \text{ or}$$
$$I_j^-(s) \cap V_j^c \neq \phi, \text{ and } \exists_j \in I_N \text{ with } I_j^+(s) \neq \phi\}.$$

For $s \in \tau^3$, each nonempty $B(s) = A(s) \cap C(s)$ consists, under some regularity conditions, of a disjoint union of smooth loops and paths with two endpoints. Each endpoint of a path in B(s) is either an endpoint of a path in $B(s^1)$ for some $s^1 \in \tau^3$, or a solution point of the NLCP, or is equal to the initial point v. The point v is only an endpoint of a path in $B(s^0)$, where $s^0 = \text{sgn } z(v)$. Notice that if an endpoint of a path in B(s) lies in C(s') with $s' \notin \tau^3$ this point is a solution to the NLCP. The set $B = \bigcup_{s \in \tau^3} B(s)$ therefore consists, under some regularity conditions and if z is continuously differentiable, of a disjoint union of piecewise smooth paths and loops. One path in B connects v with a solution x^*. The latter path is generated by the exponent-process. The other paths in B connect two solution points.

4. The interpretation of the exponent-process in strategic terms.

To become more familiar with the matter we start this section with an example concerning a noncooperative game with two

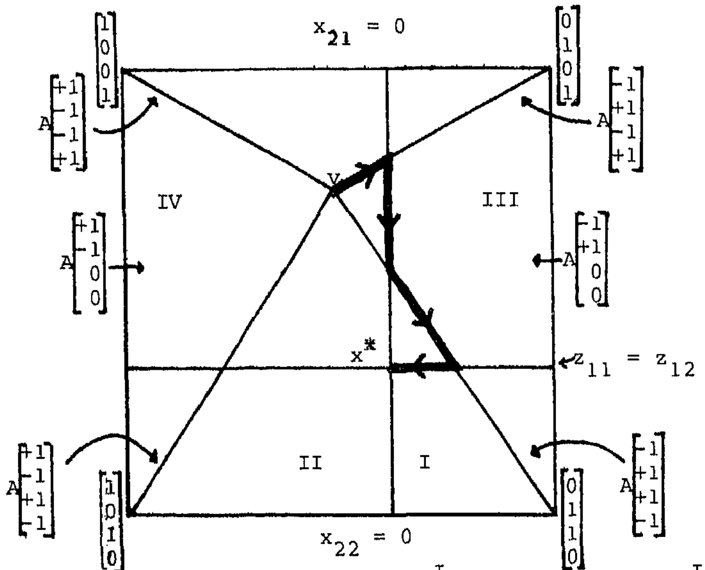

Figure 1. The areas $C((+1,-1,+1,-1)^T)$, $C((+1,-1,-1,+1)^T)$, $C((-1,+1,+1,-1)^T)$ and $C((-1,+1,-1,+1)^T)$ are denoted by I, II, III and IV, respectively. The path generated by the exponent-process is heavily drawn.

The process starts from the strategy vector v, where the second strategy of each player is profitable while their respective first strategies are unprofitable. The process then enters $A((-1,+1,-1,+1)^T)$ by increasing proportionally the probabilities of the profitable strategies of each player and by decreasing proportionally all the probabilities belonging to the unprofitable ones till a strategy vector for which player 2 isin equilibrium is generated. The process now proceeds in $A((-1,+1,0,0)^T)$ by increasing the probability of the first strategy of player 2 relatively away from the probability of player one's first strategy while keeping player 2 in equilibrium. Then the process reaches $A((-1,+1,+1,-1)^T)$ where the probability of the second strategy of player 2 becomes, relatively to the initial probability vector, equal to the probability belonging to the first strategy of player 1. Then player 2 is got out of equilibrium and the process enters

$B((-1,+1,+1,-1)^T)$ by making the second strategy of player 2 unprofitable and his first one profitable. In $B((-1,+1,+1,-1)^T)$ the process reaches a vector at which player 1 is in equilibrium. Via a curve in $B((0,0,+1,-1)^T)$ the Nash equilibrium x^* is reached.

An important feature of the exponent-process is that probabilities belonging to profitable strategies are initially increased while those of unprofitable strategies are decreased. When initially strategy (j,h) is profitable, the expected loss to player j if he plays strategy h is less than the expected loss when playing his strategy vector v_j. So, his expected profit will increase when x_{jh} becomes larger. Similar arguments can be given for the treatment of initially unprofitable strategies.

Now we present the interpretation of the exponent-process when applied to a noncooperative N-person game. From the arbitrarily chosen initial strategy vector v, the exponent-process decreases proportionally the probabilities of all the unprofitable strategies of the game with the same rate and increases for each player the probabilities belonging to the profitable strategies. For each player the amount of increase of a probability belonging to a profitable strategy initially not used is equal to the rate with which the probabilities of all his profitable strategies having positive initial probability are increased.

In general, the exponent-process generates strategy vectors x for which the probabilities of the unprofitable strategies in the game are relatively (to v) equal to each other but relatively smaller than the probabilities of all the other strategies. Moreover, for each player the probabilities of his initially not used profitable strategies are kept positive and equal to each other and equal to the rate with which the probabilities belonging to his other profitable strategies are equally increased. Finally, the probabilities correspond-

ing to the profitable strategies of a player are always kept relatively (absolutely) larger than those of all his other strategies. In principle the strategy of a player which becomes in equilibrium, i.e. for which the excess profit becomes equal to zero, is kept in equilibrium.
If this strategy was unprofitable its probability is relatively increased away from the probabilities of the unprofitable strategies in the game while if this strategy was profitable its probability is relatively decreased away from the probabilities belonging to his profitable strategies. It might happen that a player becomes in equilibrium so that he does not have any profitable strategy left. In this case the probabilities of his (nonprofitable) strategies are changed in order to keep him in equilibrium.
If for a player the probability of a strategy in equilibrium becomes relatively equal to the probabilities of the unprofitable strategies in the game, then the first probability is kept relatively equal to the latter ones and the corresponding strategy is made unprofitable. In this way a player might get out of equilibrium in which case also the strategy with the relative (absolute) largest probability is made profitable. Although it seems somewhat peculiar to disturb the equilibrium position of a player he can now only be kept in equilibrium by decreasing the probability of one of his strategies in equilibrium under the level of the unprofitable strategies in the game. But then cycling or leaving the strategy space might occur. Finally, if for a player the probability of a strategy in equilibrium becomes relatively equal to the probabilities of his profitable strategies (if any), then the first probability is kept relatively equal to the latter ones and the corresponding strategy is made profitable. The process stops when all players are in equilibrium, i.e. when no player has a profitable strategy anymore.

To conclude we want to make a short comparison between the interpretation of the sum-, product- and exponent-process as adjustment processes when applied to the problem of finding a Nash equilibrium in a noncooperative N-person game. The sum-process follows a path of strategy vectors x such that for all $j \in I_N$, $x_{jk} \geq b_j v_{jk}$ if $z_{jk}(x) = \max_{(i,h)\in I} z_{ih}(x)$ and $x_{jk} = b_j v_{jk}$ if $z_{jk}(x) \neq \max_{(i,h)\in I} z_{ih}(x)$, where $0 \leq b_j \leq 1$. Also here, this memorizing of the starting vector guarantees convergency. When we observe the sum-process more precisely we see that it increases initially the probability of the most profitable strategy in the game while the probabilities corresponding to the other strategies of that player are decreased all with the same rate. The strategy vectors of the other players are not changed. In general the sum-process follows a path of strategy vectors such that for each player the probabilities of his strategies not having maximal excess profit in the game are kept relatively equal to each other and relatively smaller than the probabilities belonging to the most profitable strategies of the game handled by that player. If a strategy of a player becomes one of the most profitable in the game, the corresponding probability is relatively increased while keeping this strategy most profitable. When for a player the probability of a most profitable strategy becomes relatively equal to the probabilities of his strategies not having maximal excess profit, the first probability is kept relatively equal to the latter probabilities and its excess profit is decreased away from the maximal excess profit. The sum-process stops when the maximal excess profit in the game is zero. The path of the product-process follows strategy vectors x for which for all $j \in I_N$, $x_{jk} \geq bv_{jk}$ if $z_{jk}(x) = \max_{(j,h)\in I(j)} z_{jh}(x)$ and $x_{jk} = bv_{jk}$ if $z_{jk}(x) \neq \max_{(j,h)\in I(j)} z_{jh}(x)$, where $0 \leq b \leq 1$.

So, the product-process initially increases the probabilities belonging to the most profitable strategies of each player, whereas the probabilities of all the other strategies in the game are decreased with the same rate. In general the product-process follows a path of strategy vectors such that all the probabilities of the strategies, not having maximal excess profit for a player, are relatively equal to each other and relatively smaller than those belonging to one's most profitable strategies. If a strategy of a certain player becomes one of his most profitable strategies, the probability of this strategy is relatively increased while keeping this strategy most profitable for him. When for a player the probability of one of his most profitable strategies becomes relatively equal to the probabilities belonging to his less profitable strategies, the first probability is kept relatively equal to the latter ones and its excess profit is decreased away from the excess profit of his most profitable strategies.

When overlooking the foregoing we see that the sum-process focusses its attention only to the most profitable strategies in the game. The corresponding probabilities are relatively increased while all the others are decreased. By successive extensions of the set of most profitable strategies in the game a solution with a maximal excess profit equal to zero is reached. Observe that during this process the probabilities corresponding to profitable (but not the most profitable) strategies might decrease. This intuitively seems to be unrealistic. The product-process considers only the most profitable strategies of each separate player. This yields a similar unrealistic interpretation as for the sum-process. The exponent-process however takes all the strategies into account. Probabilities belonging to profitable strategies are relatively increased while those corresponding to unprofitable strategies are relatively decreased. The exponent-process searches for an equilibrium by adapting all the probabilities at once and in a direct manner, i.e. till a strategy has become in equilibrium. The sum- and product-process however generate a path by only considering part of the strategies. The Nash equilibrium is reached in an indirect manner, i.e. when the set of most profitable strategies is such that the corresponding excess-profit is zero. Besides, the exponent-process possesses the most appealing strategicinterpretation. Why should only the probabilities of the most profitable strategy be increased and not those of other profitable strategies? All this makes the exponent-process to the theoretically most interesting strategy-adjustment process. Moreover, we may expect that the exponent-process converges quicker than the other processes. A simplicial algorithm to approximately follow the path of points generated by the process will be described in a subsequent paper.

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